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1994 J. Phys. A: Math. Gen. 27 757

(http://iopscience.iop.org/0305-4470/27/3/019)

J. Phys. A: Math. Gen. 27 (1994) 757–762. Printed in the UK

Tupling in three-dimensional reversible mappings

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Received 4 February 1993, in final form 14 October 1993

Abstract. Feigenbaum's parameter scaling exponents δ are calculated for $l\cdot 3^m$, $l\cdot 5^m$, $l\cdot 7^m$, $l\cdot 9^m$ sequences of periodic orbits in a class of three-dimensional non-measure-preserving reversible mappings. Some preliminary results on orbit scaling exponents for $l\cdot 3^m$ sequences are also included. Scaling exponents found so far are the same as those found for two-dimensional reversible mappings.

The presence of time-reversal symmetry (reversibility) in a dynamical system has important consequences for the dynamic behaviour (for a review see [1]). Reversible mappings of the plane have been shown to exhibit a number of qualitative and quantitative similarities with their Hamiltonian (symplectic) counterparts [1–5]. Analogous to Hamiltonian systems, repeated application of local versions of the (reversible) KAM theorems and the Poincaré–Birkhoff theorem lead to a complex picture of phase space in which structures are repeated on all scales. A particular aspect of this self-similarity known as class renormalization [6] concerns the scaling properties of periodic orbits. The infinite hierarchy of periodic orbits observed is known to play an essential role in transport models of reversible and Hamiltonian systems [7].

Period-doubling has been studied extensively in reversible area-preserving mappings [8–14], non-area-preserving reversible mappings [1,2] of the plane and polynomial mappings of the complex plane [15]. Sequences of higher-order bifurcating sequences (n-tupling with $n>2$) have also been studied in these mappings [6, 16, 17]. The scaling behaviour of higher-dimensional mappings and their associated universality classes are less well understood and study has been restricted to period doubling in four- and six-dimensional symplectic reversible mappings [18–21]. Period doubling associated with curves of orbits in three-dimensional volume-preserving reversible mappings with one integral have recently been examined [22]. In this study we construct a three-dimensional non-measure-preserving reversible mapping and study the scaling properties of sequences of odd periodic orbits in some examples of this mapping.

A mapping $T: R^m \mapsto R^m$ is called *reversible* if it can be written as the composition of two involution mappings I_1 and I_2,

$$T = I_1 \circ I_2 \tag{1}$$

and

$$I_1 \circ I_1 = I_2 \circ I_2 = Id\,. \tag{2}$$

Define

$$\begin{aligned} A_{n,i} &= \mathrm{Fix}(I_i) \cap \mathrm{Fix}(T^{2n} I_i) \qquad i = 1, 2 \\ B_n &= \mathrm{Fix}(I_2) \cap \mathrm{Fix}(T^{2n} I_1) \end{aligned} \tag{3}$$

where Fix denotes the set of invariant points of a mapping and T^n denotes the mapping T composed n times. If $r \in A_{n,i}$ ($i = 1, 2$), then r is a point on a periodic orbit of length $2n$. If $r \in B_n$, then r is a point on a periodic orbit of length $2n + 1$ [1].

Now consider a class of non-measure- and orientation-preserving two-dimensional ($m = 2$) reversible mappings given by

$$T_{2D} : x' = \frac{f_1(y) + x}{1 - xf_3(y)} \quad y' = \frac{g_1(x') + y}{1 - yg_3(x')} \tag{4}$$

with involutions

$$I_1 : x' = x \quad y' = \frac{g_1(x) - y}{1 + yg_3(x)} \qquad I_2 : x' = \frac{f_1(y) + x}{1 - xf_3(y)} \quad y' = -y \tag{5}$$

where functions g_1 and g_3 are arbitrary while f_1 and f_3 are arbitrary odd functions (this corresponds to class IV of [1]). The fixed points of the involutions ($I_i r = r$, $i = 1, 2$) form curves in the plane called symmetry lines. As mentioned above, symmetry lines have special significance because along these lines are points belonging to periodic orbits of T. Not all periodic orbits of T have a point on one or both of these lines but those that do are referred to as *symmetric* periodic orbits. The task of finding symmetric periodic orbits is thus reduced to a search along a line rather than a plane. Symmetric orbits of even period have two points on one of the symmetry lines while symmetric orbits of odd period have one point on both of the lines. Symmetric periodic orbits have return Jacobian determinant equal to ± 1 depending on whether T is orientation preserving ($+1$) or reversing (-1) and on the odd or even length of the periodic orbit. Our strategy for extending the class of mappings given by (4) into three dimensions is to first trivially extend both the involutions

$$I_1 : x' = x \quad y' = \frac{g_1(x) - y}{1 + yg_3(x)} \quad z' = -z \tag{6}$$

$$I_2 : x' = \frac{f_1(y) + x}{1 - xf_3(y)} \quad y' = -y \quad z' = z\,. \tag{7}$$

Clearly z is decoupled in these equations and to remedy this we conjugate one of the involutions, I_2 by a nonlinear invertible transformation

$$S : x' = x + \epsilon z h(y) \quad y' = y \quad z' = z + \epsilon y \tag{8}$$

where h is an arbitrary function, ϵ is a perturbation parameter (when $\epsilon = 0$ (8) is the identity and the composition of the two involutions is decoupled in z and hence effectively two-dimensional). Note that $I_2' = S^{-1} \circ I_2 \circ S$ is still an involution. We now construct the new mapping $T_{3D} : R^3 \mapsto R^3$ as

$$T_{3D} = I_1 \circ I_2' = I_1 \circ (S^{-1} \circ I_2 \circ S) \tag{9}$$

which written explicitly gives

$$T_{3D} : x' = \frac{f_1(y) + x + \epsilon z h(y)}{1 - (x + \epsilon z h(y)) f_3(y)} - \epsilon(z+2\epsilon y)h(-y) \quad y' = \frac{g_1(x') + y}{1 - yg_3(x')} \quad z' = -(z+2\epsilon y)\,. \tag{10}$$

Table 1. Parameter and function settings for the four mappings. Examples 1, 2 and 3 are given by (10) and example 4 by (12).

	Example 1	Example 2	Example 3	Example 4
g_3	0	$-c$	$-c$	—
$h(y)$	y	y	y^2	—
c	0.02	0.1	0.1	—
ϵ	0.001	0.001	0.01	0.01

Observe that Fix I_1 is one-dimensional (curve) while Fix I_2' is two-dimensional (plane). Referring to (3), this means that the set $A_{n,1}$ will, in general, be empty. The set $A_{n,2}$ will be infinite and the set B_n will be finite. This corresponds respectively to zero, infinitely many or finitely many symmetric periodic orbits. Since we wish to have a finite set of periodic orbits of a given length we only consider the last case here, i.e. symmetric periodic orbits of odd length (cf [22]).

For our first three examples we choose (following [2]) the functions f_1, f_3, g_1 and g_3 as

$$f_1(x) = \frac{-x}{1+c^2x^2} \qquad f_3(x) = \frac{c^2x^3}{c^2+x^2} \qquad g_1(x) = 2x(1-k-x) \qquad g_3(x) = \text{const}. \tag{11}$$

The parameter c represents the non-measure-preserving perturbation and is fixed, while k is a free parameter which we use to follow sequences of periodic orbits. Table 1 indicates the various choices of g_3, $h(y)$, c and ϵ for our three examples. Our fourth example is also a non-measure-preserving reversible three-dimensional mapping constructed from a different class of two-dimensional mappings (class I of [1]) in the same way

$$x' = (k-y)(1+(y'-1)(y'-1)) \quad y' = \frac{x+\epsilon(2y-k)(z+\epsilon(y-k))}{1+(y+1-k)^2} \quad z' = -z+\epsilon(k-2y). \tag{12}$$

Again k is a free parameter while ϵ is specified in table 1.

The linear stability of a periodic orbit is determined by the eigenvalues of the Jacobian matrix which satisfy

$$\lambda^3 - A\lambda^2 + B\lambda - C = 0 \qquad A = \text{tr}(J) \quad B = [\text{tr}^2(J) - \text{tr}(J^2)]/2 \quad C = \det(J) \tag{13}$$

where $\text{tr}(J)$ and $\det(J)$ denote the trace and determinant of the 3×3 return Jacobian matrix. The three coefficients A, B and C are not independent and can be reduced: for symmetric odd periodic orbits in orientation-reversing mappings $C = \det(J) = -1$ and we have one eigenvalue $\lambda = -1$ [22] which implies $A = -B$, so effectively there is just the one free coefficient (as is the case in two dimensions). The residue [23] can be written as

$$R = \tfrac{1}{4}[2 - (\text{tr}(J) + 1)]. \tag{14}$$

For $R < 0$ or $R > 1$ the orbit is unstable (hyperbolic) while for $0 < R < 1$ the orbit is stable (elliptic). For elliptic orbits we have

$$R = \sin^2(\pi\omega) \tag{15}$$

Table 2. Parameter scaling exponents $\delta(p, q)$ for various p/q-tupling sequences in *three-dimensional* reversible maps. For comparison we give the corresponding results for *two-dimensional* area-preserving reversible maps [6, 16] in the last row.

	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{9}$
Example 1	20.184	20.047	—	10.807	—	9.08
Example 2	20.1848	20.047	—	10.80	—	9.0
Example 3	20.184	20.047	—	10.807	—	
Example 4	20.1	—	30.2	—	39.2	
2D maps	20.1848	20.0478	30.257	10.8076	39.279	9.0814

Table 3. Orbit scaling exponents for $\frac{1}{3}$-tupling sequences in *three-dimensional* reversible maps (see figure 1). For comparison we give the corresponding results for *two-dimensional* area-preserving reversible maps in the last row.

	α_1	α_2	β_1	β_2
Example 1	-17.9	2.45	5.94	-31.4
Example 2	-17.9	2.4	5.94	-31.4
2D maps	-17.9	2.45	5.94	-31.4

where ω is the rotation number of a given orbit. Every periodic orbit is characterized by a rational rotation number $\omega = p/q$ (i.e $\lambda = \exp(2\pi i p/q)$) so that setting ω equal to some rational frequency in (15) gives the critical value of the residue for which periodic orbits of the same frequency are born†. The values of k corresponding to the critical value of the residue for each orbit with the appropriate frequency have been observed to form a convergent sequence for various classes of mappings [1, 15]. The convergence is asymptotically geometric with the scaling factor δ given by

$$\delta_n = \frac{k_{n-1} - k_n}{k_n - k_{n+1}} \tag{16}$$

and the limiting value of δ depends on the frequency, i.e. $\delta(p, q)$ [6, 16, 17].

We initially locate the sequences of periodic orbits $l \cdot 3^m$, $l \cdot 5^m$, $l \cdot 7^m$ and $l \cdot 9^m$ in (10) for $\epsilon = 0$ (the two-dimensional orbits) then follow these orbits as ϵ is increased to some fixed value (see table 1). For most sequences our starting point is the symmetric (elliptic) fixed point of T_{3D} (i.e. $l = 1$), however, $5 \cdot 3^m$ and $3 \cdot 5^m$ sequences were also examined in some of the examples. As k is increased periodic orbits of various frequencies are born at the fixed point and the appropriate orbits are tracked. Periodic orbits with length of order 10^6 were located using the secant method and calculations were performed in quadruple-precision VAX FORTRAN. In table 2 are the results obtained for parameter scalings of our four examples specified by (10), (12) and table 1. In table 3 we present some preliminary results on orbit scaling for $l \cdot 3^m$ sequences in two examples of the mapping (10). We calculate the distance scalings associated with three-points of each $l \cdot 3^m$ orbit that have bifurcated from the point of the $l \cdot 3^{m-1}$ orbit that lies in the symmetry plane (see figure 1 and [16]; for alternative orbit scaling exponents for two-dimensional maps refer to [6]). The two distances for the $l \cdot 3^m$ orbit are calculated at the birth of the $l \cdot 3^{m+1}$ orbit. Two distance scalings of the $l \cdot 3^m$ sequences of orbits were determined to be $\alpha = \alpha_1\alpha_2 \approx -44$ and $\beta = \beta_1\beta_2 \approx -187$. The

† The eigenvalues of non-symmetric periodic orbits in one-parameter families of reversible maps generically do not lie on the unit circle. Therefore, non-symmetric periodic orbits in such families of maps do not exhibit q-tupling for $q \neq 2$.

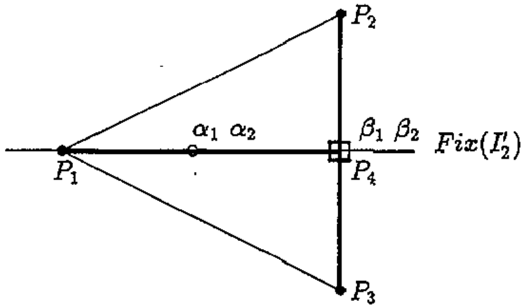

Figure 1. Schematic diagram of the distances used to calculate the orbit scaling exponents α_i and $\beta_i (i = 1, 2)$ for a $l \cdot 3^m$ orbit. Drawn are the triangle formed by three points of the $l \cdot 3^m$ orbit (full circles denoted P_j, $j = 1, \dots 3$) which has bifurcated from the $l \cdot 3^{m-1}$ orbit (open circle); the horizontal line represents the symmetry plane $\mathrm{Fix}(I_2')$. The points P_1 and P_4 (open square which is not a point in any of the orbits) are in $\mathrm{Fix}(I_2')$. The distances calculated are indicated by the heavy lines. The orbit scalings are simply the ratio of each distance of successive orbits: $\alpha_{1,2} := \overline{(P_1 - P_4)_m} / \overline{(P_1 - P_4)_{m-1}}$ and $\beta_{1,2} := \overline{(P_2 - P_3)_m} / \overline{(P_2 - P_3)_{m-1}}$. There is an alternation in the magnitude and sign of the distances so that they scale as the products $\alpha = \alpha_1 \alpha_2$ and $\beta = \beta_1 \beta_2$.

scaling exponents obtained so far appear to be the same as those found for area-preserving two-dimensional mappings. We hope to publish an extended version of this work in the near future.

Acknowledgments

G S T gratefully acknowledges the support of an Australian Postgraduate Research Award. This investigation is part of the research programme of the Australian Research Council. We also wish to thank John Roberts for his comments on the manuscript.

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